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Strong and Weak Weighted Convergence of Jacobi Series

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Given $\alpha, \beta > -1$, let $p_n(x) = p_n^{(\alpha,\beta)}(x)$, $n = 0, 1, 2, \ldots$ be the sequence of Jacobi polynomials orthonormal on $(-1, 1)$ with respect to the weight $u(x) = (1-x)^\alpha (1+x)^\beta$. Denote by $(S_N f)(x)$ the Nth partial sum of the Fourier–Jacobi series of the function f on $(-1, 1)$, so that $(S_N f)(x) = \sum_{n=0}^N a_n p_n(x)$, with $a_n = \int_{-1}^1 f(x)\, p_n(x)\, u(x)\, dx$. For fixed $p \in (1, \infty)$, we characterize the weights w such that $\lim_{N\to\infty} \int_{-1}^1 |[(S_N f)(x) - f(x)]\, w(x)|^p\, u(x)\, dx = 0$ whenever $\int_{-1}^1 |f(x)\, w(x)|^p\, dx < \infty$, the weights w such that $\lim_{N\to\infty} \sup_{\lambda>0} \lambda[\int_{E_\lambda^N} w(x)^p\, u(x)\, dx]^{1/p} = 0$ whenever $\int_{-1}^1 |f(x)\, w(x)|^p\, dx < \infty$, and the weights w such that $\lim_{N\to\infty} \sup_{\lambda>0} \lambda[\int_{E_\lambda^N} w(x)^p\, u(x)\, dx]^{1/p} = 0$ whenever $\int_0^\infty [\int_{F_\lambda} w(x)\, dx]^{1/p}\, d\lambda < \infty$; here, $E_\lambda^N = \{x \in (-1, 1); |(S_N f(x) - f(x)| > \lambda\}$ and $F_\lambda = \{x \in (-1, 1) : |f(x)| > \lambda\}$. © 1997 Academic Press

I. INTRODUCTION

Let $u(x) = u_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -1$, be a Jacobi weight and let $p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n x^n + \ldots$, $\gamma_n > 0$, $n = 0, 1, 2, \ldots$, be the corresponding sequence of (orthonormal) Jacobi polynomials

$$\int_{-1}^1 p_m(x)\, p_n(x)\, u(x)\, dx = \delta_{mn} \qquad m, n = 0, 1, 2, \ldots . \tag{1}$$

For example, $p_n^{(-1/2,-1/2)}(x) = (2/\sqrt{\pi}) \cos(n \cos^{-1} x)$, $n = 1, 2, \ldots$, and the $p_n^{(0,0)}(x)$ are the (normalized) Legendre polynomials. An extended treatment of Jacobi polynomials can be found in [20]; in particular, one finds on p. 198 the asymptotic formula

$$p_n(\cos\theta) = \sqrt{\frac{2}{\pi}}\, u(\cos\theta)^{-1/2} [\cos(M\theta + \gamma) + (n \sin\theta)^{-1}\, 0(1)] \tag{2}$$

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for $cn^{-1} \leqslant \theta \leqslant \pi - cn^{-1}$, $c > 0$ a fixed constant, $M = n + (\alpha + \beta + 1)/2$, and $\gamma = -(\alpha + 1/2)\,\pi/2$.

Given a function satisfying $\int_{-1}^{1} |f(x)|\, u(x)\, dx < \infty$, denote by $(S_N f)(x)$ the Nth partial sum of the Fourier–Jacobi series of f, namely $\sum_{n=0}^{N} a_n p_n(x)$, with $a_n = \int_{-1}^{1} f(x)\, p_n(x)\, u(x)\, dx$. The purpose of this paper is to characterize the nonnegative, measurable (weight) functions on $(-1, 1)$ such that for fixed $p \in (1, \infty)$ $S_N f$ converges to f in one of the following senses:

1. weighted strong sense

$$\lim_{N \to \infty} \int_{-1}^{1} |[(S_N f)(x) - f(x)]\, w(x)|^p\, u(x)\, dx = 0, \tag{3}$$

whenever

$$\int_{-1}^{1} |f(x)\, w(x)|^p\, u(x)\, dx < \infty;$$

2. weighted weak sense

$$\lim_{N \to \infty} \sup_{\lambda > 0} \lambda \left[\int_{E_\lambda^N} w(x)^p\, u(x)\, dx \right]^{1/p} = 0, \tag{4}$$

$E_\lambda^N = \{x \in (-1, 1) \colon |(S_N f)(x) - f(x)| > \lambda\}$, whenever

$$\int_{-1}^{1} |f(x)\, w(x)|^p\, u(x)\, dx < \infty;$$

3. weighted restricted weak sense

$$\lim_{N \to \infty} \sup_{\lambda > 0} \lambda \left[\int_{E_\lambda^N} w(x)^p\, u(x)\, dx \right]^{1/p} = 0, \tag{5}$$

$E_\lambda^N = \{x \in (-1, 1) \colon |(S_N f)(x) - f(x)| > \lambda\}$, whenever

$$\int_0^{\infty} \left[\int_{F_\lambda} w(x)^p\, u(x)\, dx \right]^{1/p} d\lambda < \infty,$$

$F_\lambda = \{x \in (-1, 1) \colon |f(x)| > \lambda\}$.

Any strong weight is a weak weight and any weak weight is a restricted weak weight. This is readily seen from the following relations between the

Lorentz $L^{p\infty}(w^p u)$ and $L^{p1}(w^p u)$ norms and the usual Lebesgue $L^p(w^p u)$ norm:

$$\|g\|_{L^{p\infty}(w^p u)} = \sup_{\lambda>0} \lambda \left[\int_{G_\lambda} w(x)^p\, u(x)\, dx\right]^{1/p} \leqslant \left[\int_{-1}^{1} |g(x)\, w(x)|^p\, u(x)\, dx\right]^{1/p}$$
$$\leqslant \int_0^\infty \left[\int_{G_\lambda} w(x)^p\, u(x)\, dx\right]^{1/p} dx = \|g\|_{L^{p1}(w^p u)},$$

$G_\lambda = \{x \in (-1, 1) \colon |g(x)| > \lambda\}$. We will show that, in fact, the weak weights are the same as the strong weights.

It will be convenient to work with the trigonometric form, $p_n(\cos\theta)$, of the Jacobi polynomials for which

$$\int_0^\pi (\cos\theta)\, p_n(\cos\theta)\, u(\theta)\, d\theta = \delta_{mm} \qquad m, n = 0, 1, 2, ...,$$

where $u(\theta) = 2^{\alpha+\beta} \sin^{2\alpha+1}(\theta/2) \cos^{2\beta+1}(\theta/2)$ and $a_n = \int_0^\pi f(\phi)\, p_n(\cos\phi) u(\phi)\, d\phi$, $f(\phi) = f(\cos\phi)$. With this notation, it is seen we have the usual Fourier cosine series when $\alpha = \beta = -1/2$. Moreover, (3), for example, becomes

$$\lim_{N\to\infty} \int_0^\pi |[(S_N f)(\theta) - f(\theta)]\, w(\theta)|^p\, u(\theta)\, d\theta = 0,$$

whenever $\int_0^\pi |f(\phi)\, w(\phi)|^p\, u(\phi)\, d\phi < \infty$. Here, $w(\theta) = w(\cos\theta)$.

Earlier weighted convergence theorems dealt with Jacobi weights $w(x) = c(1+x)^A\, (1-x)^B$ or, equivalently, $w(\theta) = c \sin^{2A+1}(\theta/2) \cos^{2B+1}(\theta/2)$. Thus, the classical 1927 result of Riesz [19] asserts (4) holds for $1 < p < \infty$ when $\alpha = \beta = -1/2$ (so that $u(\theta) = 1/2$) and $w(\theta) = 1$. The pioneering 1948 paper of Pollard [18] characterized the range of indices p that work in (4) for $\alpha, \beta \geqslant -1/2$ and $w(\theta) = 1$; for instance, in the Legendre case $\alpha = \beta = 0$, the range is $4/3 < p < 4$. Muckenhoupt proved the definitive theorem on Jacobi weights for $\alpha, \beta > -1$ in [10]. Askey [1] used such results to investigate the weighted strong convergence of Lagrange interpolation polynomials based on the zeros of Jacobi polynomials.

Badkov [3] studied the generalized Jacobi polynomials satisfying (1) with respect to u of the form

$$u(x) = H(x)(1+x)^\alpha\, (1-x)^\beta \prod_{k=1}^{m} |x - x_k|^{\gamma_k};$$

here, $\alpha, \beta, \gamma_k > -1$, $-1 < x_1 < \cdots < x_m < 1$, $H(x) > 0$ on $(-1, 1)$, and $\int_0^2 \omega(H, \delta)\, d\delta/\delta < \infty$, $\omega(H, \delta)$ being the usual modulus of continuity of H in $C([-1, 1])$. He extended Muckenhoupt's results to Fourier series in such polynomials for the generalized Jacobi weights $w(x) = C(1+x)^A (1-x)^B \prod_{k=1}^m |x-x_k|^{\Gamma_k}$; the x_k are the same as in $u(x)$ and $A, B, \Gamma_k > -1$. Nevai in a series of papers, [16], completed the above-mentioned work of Askey for generalized Jacobi polynomials.

Chanillo [4] proved the first restricted weak convergence theorem for Fourier–Legendre series when $p=4/3$ or $p=4$ and $w(\theta)=1$. The result for this weight was extended to all Fourier–Jacobi series by Guadalupe, Perez and Varona [5]; for example, when $\alpha \geqslant \beta \geqslant -1/2$ and $\alpha > -1/2$, they obtained restricted weak convergence when $p = 4(\alpha+1)/(2\alpha+3)$ or $p = 4(\alpha+1)/(2\alpha+1)$, values of p for which they show weak convergence doesn't hold.

The general weighted strong convergence problem for Fourier series ($\alpha=\beta=-1/2$) was solved by Hunt, Muckenhoupt, and Wheeden [7]. A weight w yields (4) in this case, for fixed $p \in (1, \infty)$, if and only if $w \in A_p(0, \pi)$:

$$\left[\frac{1}{|I|}\int_I w(\theta)^p \, d\theta\right]^{1/p} \left[\frac{1}{|I|}\int_I w(\theta)^{-p'} \, d\theta\right]^{1/p'} \leqslant C,$$

where, as usual, $p' = p/(p-1)$ and the constant $C > 0$ is independent of the interval $I \subset (0, \pi)$ with length $|I|$.

A standard argument involving the Banach–Steinhaus theorem shows (3) is equivalent to

$$\int_0^\pi |(X_N f)(\theta)\, w(\theta)|^p \, u(\theta)\, d\theta \leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p \, u(\phi)\, d(\phi), \tag{6}$$

(4) is equivalent to

$$\sup_{\lambda>0} \lambda \left[\int_{E_\lambda^N} w(\theta)^p \, u(\theta)\, d\theta\right]^{1/p} \leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p \, u(\phi)\, d\phi, \tag{7}$$

and (5) is equivalent

$$\sup_{\lambda>0} \lambda \left[\int_{E_\lambda^N} w(\theta)^p \, u(\theta)\, d\theta\right]^{1/p} \leqslant C \int_0^\infty \left[\int_{F_\lambda} w(\phi)^p \, u(\phi)\, d\phi\right]^{1/p} d\lambda; \tag{8}$$

here, $E_\lambda^N = \{\theta \in (0, \pi) \colon |(S_N f)(\theta)| > \lambda\}$, $F_\lambda = \{\phi \in (0, \pi) \colon |f(\phi)| > \lambda\}$ and $C > 0$ is independent of f and N.

The converse of Hölder's inequality for Lebesgue and Lorentz spaces (see [6] for the latter) readily yields

LEMMA 1. *The condition*

$$\int_0^{\pi} w(\phi)^{-p'} u(\phi)\, d\phi < \infty \tag{9}$$

is necessary and sufficient to guarantee

$$\int_0^{\pi} |f(\phi)|\, u(\phi)\, d\phi < \infty, \tag{10}$$

whenever

$$\int_0^{\pi} |f(\phi)\, w(\phi)|^p\, u(\phi)\, d\phi < \infty.$$

Similarly,

$$\|w^{-p}\|_{L^{p'\infty}(w^p u)} = \sup_{\lambda > 0} \lambda \left[\int_{E_\lambda} w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p'} < \infty, \tag{11}$$

$E_\lambda = \{\phi \in (0, \pi) : w(\phi)^{-p} > \lambda\}$, *is necessary and sufficient to have* (10) *whenever*

$$\int_0^{\infty} \left[\int_{F_\lambda} w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} d\lambda < \infty,$$

$F_\lambda = \{\phi \in (0, \pi) : |f(\phi)| > \lambda\}$.

The condition (10) is required, of course, in order that the Fourier–Jacobi series of f be defined. Further, the constant function $p_0(\theta)$ (and hence $S_N f$ for all N) belongs to $L^p(w^p u)$ (equivalently to $L^{p\infty}(w^p u)$) if and only if

$$\int_0^{\pi} w(\theta)^p\, u(\theta)\, d\theta < \infty. \tag{12}$$

With these facts in mind, we call weights satisfying (9) and (12) S-admissible, while weights for which (11) and (12) hold will be said to be RW-admissible.

We can now state our main results.

THEOREM 2. *Fix* $\alpha, \beta > -1$ *and* $p \in (1, \infty)$. *Let* $u(\theta) = u_{\alpha,\beta}(\theta) = 2^{\alpha+\beta} \sin^{2\alpha+1}(\theta/2) \cos^{2\beta+1}(\theta/2)$ *and, given* f *with* $\int_0^{\pi} |f(\phi)|\, u(\phi)\, d\phi < \infty$, *let* $(S_N f)(\theta) = \sum_{n=0}^{N} a_n p_n(\cos\theta)$, $a_n = \int_0^{\pi} f(\phi)\, p_n(\phi)\, u(\phi)\, d\phi$, *be the* N*th partial sum of the Fourier–Jacobi series of* f. *Then, given an* S*-admissible weight* w *on* $(0, \pi)$, *the following are equivalent*:

(a) $$\lim_{N\to\infty}\int_0^\pi |[(S_N f)(\theta)-f(\theta)]\,w(\theta)|^p\,u(\theta)\,d\theta=0,$$

whenever $\int_0^\pi |f(\phi)\,w(\phi)|^p\,u(\phi)\,d\phi<\infty$;

(b) $$\lim_{N\to\infty}\sup_{\lambda>0}\lambda\left[\int_{E_\lambda^N} w(\theta)^p\,u(\theta)\,d\theta\right]^{1/p}=0,$$

$E_N^\lambda=\{\phi\in(0,\pi)\colon |(S_N f)(\phi)-f(\phi)|>\lambda\}$, *whenever* $\int_0^\pi |f(\phi)\,w(\phi)|^p\,u(\phi)\,d\phi<\infty$;

(c) *There exists* $C>0$ *such that*

$$\left[\frac{1}{|I|}\int_I w(\theta)^p\,u(\theta)\,d\theta\right]^{1/p}\left[\frac{1}{|I|}\int_I w(\theta)^{-p'}\,u(\theta)\,d\theta\right]^{1/p'}\leqslant C \tag{13}$$

and

$$\left[\frac{1}{|I|}\int_I w(\theta)^p\,u(\theta)^{1-p/2}\,d\theta\right]^{1/2}\left[\frac{1}{|I|}\int_I w(\theta)^{-p'}\,u(\theta)^{1-p'/2}\,d\theta\right]^{1/p'}\leqslant C \tag{14}$$

for all intervals $I\subset(0,\pi)$. *When* $\alpha,\beta\geqslant -1/2$ (14) *implies* (13), *so* (14) *alone is required in that case.*

THEOREM 3. *Let* α, β, u, f, *and* $S_N f$ *be as in Theorem* 2. *Then, in order that the RW-admissible weight* w *on* $(0,\pi)$ *satisfy*

$$\lim_{N\to\infty}\sup_{\lambda>0}\lambda\left[\int_{E_\lambda^N} w(\theta)^p\,u(\theta)\,d\theta\right]^{1/p}=0,$$

$E_\lambda=\{\theta\in(0,\pi)\colon |(S_N f)(\theta)-f(\theta)|>\lambda\}$, *whenever* $\int_0^\infty [\int_{F_\lambda} w(\phi)^p\,u(\phi)\,d\phi]^{1/p}\,d\lambda<\infty$, $F_\lambda=\{\phi\in(0,\pi)\colon |f(\phi)|>\lambda\}$, *it is necessary and sufficient that there exist* $C>0$, *independent of all intervals* $I\subset(0,\pi)$ *and measurable sets* $E\subset(0,\pi)$, *related to* I *as specified below, such that*

$$\frac{\int_E u(\phi)\,d\phi}{\int_I u(\phi)\,d\phi}\leqslant C\left[\frac{\int_E w(\phi)^p\,u(\phi)\,d\phi}{\int_I w(\phi)^p\,u(\phi)\,d\phi}\right]^{1/p}\qquad \forall E\subset I \tag{15}$$

and

$$\frac{\int_E u(\phi)^{1/2}\,d\phi}{\int_0^\theta u(\phi)^{1/2}\,d\phi}\leqslant C\left[\frac{\int_E w(\phi)^p\,u(\phi)\,d\phi}{\int_0^\theta w(\phi)^p\,u(\phi)\,d\phi}\right]^{1/p}\qquad \forall E\subset(0,\theta) \tag{16}$$

$$\frac{\int_E u(\phi)^{1/2}\,d\phi}{u(\theta)^{1/2}\sin(\phi/2)}\leqslant C\left[\frac{\int_E w(\phi)^p\,u(\phi)\,d\phi}{\int_0^\theta w(\phi)^p\,u(\phi)\,d\phi}\right]^{1/p}\qquad \forall E\subset(\theta,\pi/2), \tag{17}$$

when $0<\theta<\phi/2$, *and*

$$\frac{\int_E u(\phi)^{1/2}\,d\phi}{\int_0^\pi u(\phi)^{1/2}\,d\phi}\leqslant C\left[\frac{\int_E w(\phi)^p\,u(\phi)\,d\phi}{\int_\theta^\pi w(\phi)^p\,u(\phi)\,d\phi}\right]^{1/p}\qquad \forall E\subset(\theta,\pi) \tag{18}$$

$$\frac{\int_E u(\phi)^{1/2}\,d\phi}{u(\theta)^{1/2}\cos(\phi/2)}\leqslant C\left[\frac{\int_E w(\phi)^p\,u(\phi)\,d\phi}{\int_\theta^\pi w(\phi)^p\,u(\phi)^p\,u\,d\phi}\right]^{1/p}\qquad \forall E\subset\left(\frac{\pi}{2},\theta\right), \tag{19}$$

when $\pi/2<\theta<\pi$.

A crucial element in the proof of a weighted convergence theorem is an alternative to the formula of Christoffel–Darboux for the Dirichlet kernel, K_N, due to Pollard [18]:

$$\begin{aligned}K_N(\theta,\phi)&=\sum_{n=0}^{N} p_n(\cos\theta)\,p_n(\cos\phi)\\&=\alpha_N h_1(\theta,\phi,N)-\beta_N[h_2(\theta,\phi,N)+h_3(\theta,\phi,N)],\end{aligned}\tag{20}$$

in which $\lim_{N\to\infty}\alpha_N=\lim_{N\to\infty}\beta_N=1$, $h_1(\theta,\phi,N)=P_{N+1}(\cos\theta)\,p_{N+1}(\cos\phi)$,

$$h_2(\theta,\phi,N)=\frac{P_{N+1}(\cos\theta)\,q_N(\cos\phi)}{2\sin((\theta+\phi)/2)\sin((\theta-\phi)/2)}$$

and

$$h_3(\theta,\phi,N)=h_2(\phi,\theta,N),$$

with $q_N(\cos\phi)=\sin^2\phi\,p_N^{(\alpha+1,\beta+1)}(\cos\phi)$. To use (20) we will rely on the fact that [20, p. 169]

$$p_N(\cos\psi)=b_N(\psi)\,s_N(\psi)^{-(\alpha+1/2)}c_N(\psi)^{-\beta+1/2)},\tag{21}$$

where $s_N(\psi)=\sin(\psi/2)+1/N$, $c_N(\psi)=\cos(\psi/2)+1/N$ and $|b_N(\psi)|\leqslant C$ for all ψ and N.

We also require a new estimate, proved in [9], for the kernel, $P^{(\alpha,\beta)}(r,\theta,\phi)$, of the Poisson integral of f

$$f(r,\theta)=\sum_{n=0}^{\infty} a_n r^n p_n(\cos\theta)=\int_0^\pi p^{(\alpha,\beta)}(r,\theta,\phi)\,f(\phi)\,u(\phi)\,d\phi,$$

$a_n=\int_0^\pi f(\psi)\,p_n(\psi)\,u(\phi)\,d\psi$.

THEOREM 4. *Let $\rho = r^{1/2}$. Then, for $1/2 < r < 1$,*

$$P^{(\alpha,\beta)}(r,\theta,\phi) \approx P(\rho,\theta,\phi)\left[\frac{1}{(1-2\rho\cos(\theta+\phi)+\rho^2)^{\alpha+1/2}} + \frac{1}{(1-2\rho\cos(\theta-\phi)+\rho^2)^{\beta+1/2}}\right], \qquad (22)$$

in the sense that each side is dominated by a constant multiple of the other. Here, $P(\rho,\theta,\phi) = (1-\rho^2)/(1-2\rho\cos(\theta-\phi)+\rho^2)$ is the classical Poisson kernel.

The proofs of the necessity of the conditions in Theorems 3 and 4 use the following special case of Theorem 2 in [14]; see also [15].

LEMMA 5. *Let f be a Lebesgue-measurable function on $(0,\pi)$. Then,*

$$\int_0^\pi u(\theta)^{1/2}\,|f(\theta)|\,d\theta \leqslant \sqrt{2\pi}\,\lim_n \int_0^\pi |p_n(\cos\theta)\,f(\theta)|\,u(\theta)\,d\theta. \qquad (23)$$

II. AUXILIARY OPERATORS

The proof of the sufficiency of (13) and (14) involves a singular integral operator related to the Hilbert transformation, namely

$$(Hf)(\theta) = (P)\int_0^\pi \frac{f(\phi)}{\sin((\theta-\phi)/2)}\,d\phi,$$

as well as the Stieltjes operators

$$(S^1 f)(\theta) = \int_0^\pi \frac{f(\phi)}{\sin(\theta/2)+\sin(\phi/2)}\,d\phi$$

and

$$(S^2 f)(\theta) = \int_0^\pi \frac{f(\phi)}{\cos(\theta/2)+\cos(\phi/2)}\,d\phi.$$

It is well-known [7] that for fixed $p \in (1,\infty)$

$$\int_0^\pi |(Hf)(\theta)\,w(\theta)|^p\,d\theta \leqslant C\int_0^\pi |f(\phi)\,w(\phi)|^p\,d\phi \qquad (24)$$

if and only if $w \in A_p(0,\pi)$. Moreover, LEMMA 6. *The condition $w \in A_p(0, \pi)$ guarantees*

$$\int_0^\pi |[(S^1 + S^2) f](\theta)\, w(\theta)|^p \, d\theta \leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p \, d\phi. \tag{25}$$

Proof. Given $f \geqslant 0$, there holds

$$\pi(S^1 f)(\theta) \leqslant \frac{1}{\theta} \int_0^\theta f(\phi)\, d\phi + \int_\theta^\pi f(\phi) \frac{d\phi}{\phi} = (Pf)(\theta) + (Qf)(\theta).$$

Now,

$$(Pf)(\theta) \leqslant (Mf)(\theta) = \sup_{\theta \in I \subset (o, \pi)} \frac{1}{|I|} \int_I |f(\phi)| \, d\phi,$$

so, in view of [11],

$$\int_0^\pi |(Pf)(\theta)\, w(\theta)|^p \, d\theta \leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p \, d\phi$$

when $w \in A_p(0, \pi)$. Again, the operator Q being the dual of P, we'll have

$$\int_0^\pi |(Qf)(\theta)\, w(\theta)|^p \, d\theta \leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p \, d\phi$$

if and only if

$$\int_0^\pi |(Pf)(\theta)\, w(\theta)^{-1}|^{p'} \, d\theta \leqslant C \int_0^\pi |f(\phi)\, w(\phi)^{-1}|^{p'} \, d\phi.$$

But, the latter is true if $w^{-1} \in A_{p'}(0, \pi)$, which is equivalent to $w \in A_p(0, \pi)$. Hence, when $w \in A_p(0, \pi)$,

$$\int_0^\pi |(S^1 f)(\theta)\, w(\theta)|^p \, d\theta \leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p \, d\phi.$$

Letting $g(\phi) = f(\pi - \phi)$ and observing that $w(\pi - \theta) \in A_p(0, \pi)$ if and only if $w(\theta) \in A_p(0, \pi)$, we get

$$\begin{aligned}\int_0^\pi |(S^2 f)(\theta)\, w(\theta)|^p \, d\theta &= \int_0^\pi |(S^1 g)(\pi - \theta)\, w(\pi - \theta)|^p \, d\theta \\ &\leqslant C \int_0^\pi |g(\phi)\, w(\pi - \phi)|^p \, d\phi \\ &\leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p \, d\phi. \quad \blacksquare\end{aligned}$$

The proof of Theorem 3 needs mapping properties of the maximal operator

$$(M_u f)(\theta) = \sup_{\theta \in I \subset (0, \pi)} = \frac{\int_I f(\phi)\, u(\phi)\, d\phi}{\int_I u(\phi)\, d\phi}$$

and the Hardy operators

$$(P_1 f)(\theta) = (\sin(\theta/2))^{-1}\, u(\theta)^{-1/2} \int_0^\theta f(\phi)\, u(\phi)^{1/2}\, d\phi$$

$$(Q_1 f)(\theta) = u(\theta)^{-1/2} \int_\theta^{\pi/2} f(\phi)\, u(\phi)^{1/2} \frac{d\theta}{\sin(\phi/2)} \qquad 0 < \theta < \pi/2$$

and

$$(P_2 f)(\theta) = (\cos(\theta/2))^{-1}\, u(\theta)^{-1/2} \int_\theta^\pi f(\phi)\, u(\phi)^{1/2}\, d\phi$$

$$(Q_2 f)(\theta) = u(\theta)^{-1/2} \int_{\pi/2}^\theta f(\phi)\, u(\phi)^{1/2} \frac{d\phi}{\cos(\phi/2)} \qquad \pi/2 < \theta < \pi.$$

The methods of [18, Proposition 1] can be easily adapted to show there exists $C > 0$, independent of f, such that

$$\sup_{\lambda > 0} \lambda \left[\int_{E_\lambda} w(\theta)^p\, u(\theta)\, d\theta \right]^{1/p} \leqslant C \int_0^\infty \left[\int_{F_\lambda} w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} d\lambda,$$

$E_\lambda = \{\theta \in (0, \pi) \colon (M_u f)(\theta) > \lambda\}$, $F_\lambda = \{\phi \in (0, \pi) \colon |f(\phi)| > \lambda\}$, if and only if (15) holds.

Concerning the Hardy operators we have the following result obtained in a collaboration with Bloom and Stepanov.

LEMMA 7. *Suppose the weight* $w \not\equiv 0$ *satisfies* (12) *and*

$$\int_0^\theta w(\phi)^p\, u(\phi)\, d\phi \approx \int_\theta^{2\theta} w(\phi)^p\, u(\phi)\, d\phi \qquad 0 < \theta < \pi/2. \tag{26}$$

Then, there exists $C > 0$, *independent of f, such that*

$$\sup_{\lambda > 0} \lambda \left[\int_{E_\lambda} w(\theta)^p\, u(\theta)\, d\theta \right]^{1/p} \leqslant C \int_0^\infty \left[\int_{F_\lambda} w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} d\lambda, \tag{27}$$

$E_\lambda = \{\theta \in (0, \pi/2): |(Tf)(\theta)| > \lambda\}$, $F_\lambda = \{\phi \in (0, \pi/2): |f(\phi)| > \lambda\}$, *if and only if* (16) *holds when* $T = P_1$ *and* (17) *holds when* $T = Q_1$.

Again, if $w \not\equiv 0$ *satisfies* (12) *and*

$$\int_0^\pi w(\phi)^p\, u(\phi)\, d\phi \approx \int_{2\theta-\pi}^{\theta} w(\phi)^p\, u(\phi)\, d\phi \qquad \pi/2 < \theta < \pi,$$

then, one has (27) *if and only if* (18) *when* $T = P_2$ *and* (27) *if and only if* (19) *when* $T = Q_2$; *in* E_λ *and* F_λ *we have* $\theta, \phi \in (\pi/2, \pi)$, *in this case.*

Proof. The proofs of the four criteria are quite similar. We give the details for Q_1. To begin, observe that (27) is equivalent to the existence of $C > 0$, independent of measurable $E \subset (0, \pi/2)$ and $\lambda > 0$, such that

$$\lambda \left[\int_{F_\lambda} w(\theta)^p\, u(\theta)\, d\theta \right]^{1/p} \leqslant C \left[\int_E w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p}, \tag{28}$$

$F_\lambda = \{\theta \in (0, \pi/2): (Q_1 \chi_E)(\theta) > \lambda\}$. See [6].

Since $w \not\equiv 0$, $0 < \int_{\pi/4}^{\pi/2} w(\phi)^p\, u(\phi)\, d\phi < \infty$, because of (26).

Thus, for $\pi/4 < \theta < \pi/2$, (17) amount to

$$\int_E u(\theta)^{1/2} \frac{d\theta}{\sin(\theta/2)} \leqslant C \left[\int_E w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} \qquad E \subset (\theta, \pi/2). \tag{29}$$

But, when $\pi/4 < \theta < \pi/2$ and $E \subset (\theta, \pi/2)$,

$$(Q_1 \chi_E)(\theta) \geqslant c \int_E u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)},$$

for some $c > 0$ independent of E. Thus, taking $\lambda = c \int_E u(\phi)^{1/2}\, d\phi/\sin(\phi/2)$ in (28) gives

$$c \int_E u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)} \left[\int_{\pi/4}^{\pi/2} w(\theta)^p\, u(\theta)\, d\theta \right]^{1/p} \leqslant C \left[\int_E w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p}$$

and (29) follows.

Suppose, then, $0 < \theta < \pi/4$. If $\theta < \phi < 2\theta$ and $E \subset (\theta, \pi/2)$,

$$(Q_1 \chi_E)(\phi) \geqslant C^{-1}\, u(\phi)^{-1/2} \int_E u(\psi)^{1/2} \frac{d\psi}{\sin(\psi/2)},$$

for some $C > 0$ independent of E, so $(Q_1\chi_E)(\phi) > \lambda$ whenever

$$u(\phi)^{-1/2} > \frac{C\lambda}{\int_E u(\psi)^{1/2}\, d\psi/\sin(\psi/2)}. \tag{30}$$

But, there exists a constant $C_1 > 0$ so that (30) always holds with $\lambda = (\int_E u(\psi)^{1/2}\, d\psi/\sin(\psi/2))/C_1 u(\theta)^{1/2}$ and $\theta < \phi < 2\theta$. Hence, from (28),

$$\begin{aligned} u(\theta)^{-1/2} \int_E u(\psi)^{1/2} \frac{d\psi}{\sin(\psi/2)} \left[\int_\theta^{2\theta} w(\phi)^p\, u(\phi)\, d\theta \right]^{1/p} \\ \leqslant C \left[\int_E w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} \end{aligned}$$

and we obtain (17) by (26).

We next show (17) implies (28). Now, for some $C > 0$, independent of E,

$$(Q_1\chi_E)(\theta) \geqslant C^{-1} u(\theta)^{-1/2} \int_{E \cap (\theta, \pi/2)} u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)},$$

so,

$$\begin{aligned} F_\lambda &= \{\theta \in (0, \pi/2) : (Q_1\chi_E)(\theta) > \lambda\} \\ &\subset \left\{ \theta \in (0, \pi/2) : u(\theta)^{-1/2} \int_{E \cap (\theta, \pi/2)} u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)} > C\lambda \right\} \\ &\subset \left\{ \theta \in (0, \pi/2) : \left[\int_{E \cap (\theta, \pi/2)} w(\phi)^p\, u(\phi)\, d\phi \Big/ \int_0^\theta w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} > C_1 \lambda \right\} \\ &\qquad \text{by (17)} \\ &\subset \left\{ \theta \in (0, \pi/2) : \lambda \left[\int_0^\theta w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} \leqslant C \left[\int_E w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} \right\} = G_\lambda. \end{aligned}$$

Thus,

$$\begin{aligned} \lambda \left[\int_{F_\lambda} w(\theta)^p\, u(\theta)\, d\theta \right]^{1/p} &\leqslant \lambda \left[\int_{G_\lambda} w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} \\ &\leqslant \lambda \left[\int_0^{\mathrm{lub}\, G_\lambda} w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} \\ &\leqslant C \left[\int_E w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p}. \quad \blacksquare \end{aligned}$$

III. PROOF OF THEOREM 2

Since (7) holds whenever (8) does, we need only show that (13) and (14) imply (7) and that (8) yields (13) and (14). To prove the former it is enough to consider f such that $\int_0^\pi |f(\phi)\, w(\phi)|^p\, u(\phi)\, d\phi = 1$. By (20),

$$|(S_N f)(\theta)| \leqslant C \sum_{k=1}^{3} \left| \int_0^\pi h_k(\theta, \phi, N)\, f(\phi)\, u(\phi)\, d\phi \right| = C \sum_{k=1}^{3} J_k(f, \theta, N). \tag{31}$$

The estimate (21) for p_{N+1} and Hölder's inequality gives

$$\begin{aligned} J_1(f, \theta, N) &= \left| p_{N+1}(\cos\theta) \int_0^\pi p_{N+1}(\cos\phi)\, f(\phi)\, u(\phi)\, d\phi \right| \\ &\leqslant C s_N(\theta)^{-(\alpha+1/2)}\, c_N(\theta)^{-(\beta+1/2)} \int_0^\pi s_N(\phi)^{-(+1/2)} \\ &\quad \times c_N(\phi)^{-(\beta+1/2)}\, |f(\phi)|\, u(\phi)\, d\phi \\ &\leqslant C\,[1+u(\theta)^{-1/2}] \int_0^\pi [1+u(\phi)^{-1/2}]\, |f(\phi)|\, u(\phi)\, d\phi \\ &\leqslant C[1+u(\theta)^{-1/2}] \left[\int_0^\pi w(\theta)^{-p'}\, [1+w(\theta)^{-1/2}]^{p'}\, u(\theta)\, d\theta \right]^{1/p'}, \end{aligned}$$

so,

$$\begin{aligned} &\int_0^\pi [J_1(f, \theta, N)\, w(\theta)]^p\, u(\theta)\, d\theta \\ &\qquad \leqslant C \left(\int_0^\pi w(\theta)^p\, [1+u(\theta)^{-1/2}]^p\, u(\theta)\, d\theta \right. \\ &\qquad\qquad \left. \times \left(\int_0^\pi w(\theta)^{-p'}\, [1+u(\theta)^{-1/2}]^{p'}\, u(\theta)\, d\theta \right)^{p-1} \right) \\ &\qquad \leqslant C, \end{aligned}$$

by (13) and (14).

Next, from (21), $J_2(f, \theta, N)$ is less than a constant times

$$\begin{aligned} &s_N(\theta)^{-(\alpha+1/2)}\, c_N(\theta)^{-(\beta+1/2)} \\ &\quad \times \left| \int_0^\pi \frac{b_{N+1}(\phi)\, \sin^2\phi f(\phi)\, s_N(\phi)^{-(\alpha+3/2)}\, c_N(\phi)^{-(\beta+3/2)}\, u(\phi)\, d\phi}{\sin((\theta+\phi)/2)\, \sin((\theta-\phi)/2)} \right|, \end{aligned}$$

where $|b_{N+1}(\phi)| \leqslant C$, $0<\phi<\pi$. But,

$$\frac{\sin\phi}{\sin((\theta+\phi)/2)\sin((\theta-\phi)/2)} = \frac{1}{\sin((\theta-\phi)/2)} + \left[\frac{\sin\phi}{\sin((\theta+\phi)/2)} - 1\right] \times \frac{1}{\sin((\theta-\phi)/2)}, = \frac{1}{\sin((\theta-\phi)/2)} + R(\theta,\phi),$$

with

$$R(\theta,\phi) = \begin{bmatrix} 0\left[\dfrac{1}{\sin(\theta/2)+\sin(\phi/2)}\right] & 0<\theta<\pi/2 \\ 0\left[\dfrac{1}{\cos(\theta/2)+\cos(\phi/2)}\right] & \pi/2<\theta<\pi. \end{bmatrix}$$

Thus, setting $g(\phi) = b_{N+1}(\phi)\sin\phi\, f(\phi)\, s_N(\phi)^{-(\alpha+3/2)}\, c_N(\phi)^{-(\beta+3/2)}\, u(\phi)$, we have

$$J_2(f,\theta,N) \leqslant C[\,|(Hg)(\theta)| + [(S^1+S^2)(|g|)](\theta)]\, s_N(\theta)^{-(\alpha+1/2)} \times c_N(\theta)^{-(\beta+1/2)}. \tag{32}$$

When $\alpha, \beta \geqslant -1/2$, it follows from (32) that

$$\int_0^\pi [J_2(f,\theta,N)\, w(\theta)]^p\, u(\theta)\, d\theta \leqslant C\int_0^\pi |[\,|(Hg)(\theta)| + [(S^1+S^2)(|g|)](\theta)\, w(\theta)]|^p\, u(\theta)^{1-p/2}\, d\theta \leqslant C\int_0^\pi |g(\phi)\, w(\phi)|^p\, u(\phi)^{1-p/2}\, d\phi \leqslant C\int_0^\pi |f(\phi)\, w(\phi)|^p\, u(\phi)\, d\phi,$$

by (13), in view of (24) and (25).

We illustrate the case $\min[\alpha, \beta] < -1/2$ by $\alpha < -1/2 \leqslant \beta$. Here, the preceding considerations yield $\int_0^\pi [J_2(f, \theta, N)\, w(\theta)]^p\, u(\theta)\, d\theta$ dominated by a constant multiple of

$$\int_0^\pi |[|(Hg)(\theta) + [(S^1 + S^2)(|g|)](\theta)\, w(\theta)\, u(\theta)^{1/p}\, S_N(\theta)^{-(\alpha+1/2)} \times c_N(\theta)^{-(\beta+1/2)}|]^p\, d\theta$$

$$\leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p\, u(\phi)\, d\phi,$$

provided

$$w(\theta)\, u(\theta)^{1/p}\, s_N(\theta)^{-(\alpha+1/2)}\, c_N(\theta)^{-(\beta+1/2)} \in A_p(0, \pi). \tag{33}$$

In proving (13) and (14) imply (33) for a given N (with constant independent of N), only intervals of the form $I = (0, b)$, $1/N < b < \pi/2$, offer any difficulty. For such I, (33) is equivalent to

$$\left[\frac{1}{|J|} \int_J w(\theta)^p\, u(\theta)\, s_N(\theta)^{-(\alpha+1/2)\,p}\, d\theta\right]^{1/p} \times \left[\frac{1}{|I|} \int_I w(\theta)^{-p'}\, u(\theta)^{1-p'}\, s_N(\theta)^{-(\alpha+1/2)\,p'}\, d\theta\right]^{1/p'} \leqslant C,$$

with $J = (0, 1/N)$, $(1/N, b)$. This inequality holds when $J = (0, 1/N)$ by (14) and when $J = (1/N, b)$ by (13).

The proof that

$$\int_0^\pi [J_3(f, \theta, N)\, w(\theta))]^p\, u(\theta)\, d\theta \leqslant C \int_0^\pi |f(\phi)\, w(\phi)|^p\, u(\phi)\, d\phi$$

is similar to the one for $J_2(f, \theta, N)$.

We now show (8) implies (13) and (14). A simple argument reduces considerations to intervals $I \subset (0, \pi)$ of the forms

(i) $I = (0, b)$ $\quad b < c\pi/2$

(ii) $I = (a, b)$ $\quad b - a < \min[a, \pi - b]$

(iii) $I = (a, \pi)$ $\quad a > \pi - c\pi/2,$

for any fixed c, $0 < c < 1$, a value of which will be specified later. We only look at cases (i) and (ii).

Given (8), there holds

$$\sup_{\lambda>0} \lambda \left[\int_{E_\lambda^r} w(\theta)^p \, u(\theta) \, d\theta \right]^{1/p} \leqslant C \left[\int_0^\pi |f(\phi) \, w(\phi)|^p \, u(\phi) \, d\phi \right]^{1/p},$$

$E_\lambda^r = \{\theta \in (0, \pi) \colon |f(r, \theta)| > \lambda\}$, with $C>0$ independent of f and $r \in (0, 1)$, since

$$f(r, \theta) = \sum_{N=0}^{\infty} (r^N - r^{N+1})(S_N f)(\theta) \qquad 0<r<1.$$

Letting $1 - r^{1/2} = |I|/6$, (22) yields

$$f(r, \theta) \geqslant \frac{c \int_I f(\phi) \, u(\phi) \, d\phi}{\int_I u(\phi) \, d\phi} \qquad f \geqslant 0, \quad \theta \in I, \tag{34}$$

whence (8) implies (13) by a standard argument, [12].

But (14) (as well as (13)) is equivalent to $A_p(0, \pi)$, if I satisfies (ii). We note in passing that a simple consequence of this is that for fixed A and B

$$\int_{3I} w(\theta)^p \sin^A \frac{\theta}{2} \cos^B \frac{\theta}{2} \, d\theta \leqslant D \int_I w(\theta)^p \sin^A \frac{\theta}{2} \cos^B \frac{\theta}{2} \, d\theta, \tag{35}$$

whenever $3I = 3(x_I - |I|/2, x_I + |I|/2 = (x_I - 3|I|/2, x_I + 3\,|I|/2)$ is of type (ii).

To obtain (14) in case (i), we show (8) implies

$$\left[\int_\theta^{2\theta} w(\phi)^p \, u(\phi)^{1-p/2} \, (\sin \phi)^{-p} \, d\phi \right]^{1/p} \times \left[\int_0^{c\theta} w(\phi)^{-p'} \, u(\phi)^{1-p'/2} \, d\phi \right]^{1/p'} \leqslant C \tag{36}$$

and observe that, by a similar argument, (8) also ensures

$$\left[\int_\theta^{c\theta} w(\phi)^p \, u(\phi)^{1-p/2} \, d\phi \right]^{1/r} \times \left[\int_\theta^{2\theta} w(\phi)^{-p'} \, u(\phi)^{1-p'/2} \, (\sin \phi)^{-p'} \, d\phi \right]^{1/p'} C. \tag{37}$$

Proceeding as in [2, pp. 21–22], we then multiply (36) by (37) and use Hölder's inequality to dominate $\theta^{-1} \approx \int_\theta^{2\theta} (\sin \phi)^{-2} \, d\phi$ by the product of the integrals over $(\theta, 2\theta)$ and thus obtain (14).

Fix $\theta \in (0, \pi/4)$ and consider functions $0 \leqslant f = f \cdot \chi_{(0, c\theta)}$, where $c \in (0, 1/2)$ will be determined below. Since $p_{N+1}(\cos\phi)\, p_{N+1}(\cos\psi)$ is the kernel of $S_{N+1} - S_N$, the inequality (8) gives

$$\sup_{\lambda > 0} \lambda \left[\int_{E_N^\lambda} w(\phi)^p\, u(\phi)\, d\phi \right]^{1/p} \leqslant C \left[\int_0^{c\theta} |f(\psi)\, w(\psi)|^p\, u(\psi)\, d\psi \right]^{1/p}, \tag{38}$$

$E_N^\lambda = \{\phi \in (0, \pi) : |(T_N f)(\phi)| > \lambda\}$, where

$$(T_N f)(\phi) = \int_0^{c\theta} (h_2 + h_3)(\phi, \psi, N)\, (\psi)\, u(\psi)\, d\psi. \tag{39}$$

Set

$$f_N(\psi) = \operatorname{sgn}[\, p_{N+1}(\cos\psi)]\, f(\psi). \tag{40}$$

Then, for $\theta \leqslant \phi \leqslant \pi/2$,

$$\begin{aligned} |(T_N f_N)(\phi)| &\geqslant \int_0^{c\theta} \frac{|q_N(\cos\phi)\, P_{N+1}(\cos\psi)\, f(\psi)|\, u(\psi)}{|\sin((\phi+\psi)/2)\, \sin((\phi-\psi)/2)|}\, d\psi \\ &\quad - \int_0^{c\theta} \frac{|P_{N+1}(\cos\phi)\, q_N(\cos\psi)\, f(\psi)|\, u(\psi)}{|\sin((\phi+\psi)/2)\, \sin((\phi-\psi)/2)|}\, d\psi \\ &\geqslant A_N(\phi) - B_N(\phi). \end{aligned}$$

We recall here (2)

$$\begin{aligned} p_N^{(\alpha+1, \beta+1)}(\cos\phi) &= \sqrt{\frac{2}{\pi}}\, u(\phi)^{-1/2}\, (\sin\phi)^{-1} \\ &\quad \times [\cos(M\phi + \gamma) + (N \sin\phi)^{-1}\, 0(1)], \end{aligned}$$

$cN^{-1} \leqslant \phi \leqslant \pi - cN^{-1}$, with $M = N + (\alpha + \beta + 3)/2$, $\gamma = -(\alpha + 3/2)\, \pi/2$.

When N is large (in particular, $N > 1/\theta$) it is possible to get pairwise disjoint open intervals $I_1, ..., I_{\ell_0}$, $I_\ell = I_\ell(N)$, $\ell_0 = \ell_0(N)$, contained in $(\theta, 2\theta)$, such that

$$|\cos(M\phi + \gamma) + (N \sin\phi)^{-1}\, 0(1)| > 1/3 \qquad \text{on} \quad J = \bigcup_{\ell=1}^{\ell_0} I_\ell$$

and

$$\bigcup_{\ell=1}^{\ell_0} 3I_\ell \supset (\theta, 2\theta) \qquad 3I_\ell \text{ satisfies (ii).}$$

Indeed, for sufficiently large N, (2) together with (23), yields, in addition,

$$A_N(\phi) \geqslant k \int_0^{c\theta} (\sin\phi)^{-1} u(\phi)^{-1/2} |f(\psi)| u(\psi)\, d\psi \qquad k = 1/4\pi$$

for $\phi \in J$. Again, by (21), we always have

$$B_N(\phi) \leqslant K \int_0^{c\theta} \frac{s_N(\phi)^{-(\alpha+1/2)} c_N(\phi)^{-(\beta+1/2)} \sin\psi\, |f(\psi)|\, u(\psi)^{1/2}}{\sin^2\phi}\, d\psi,$$

$\theta < \phi < \pi/2$. On $J = J(N)$, for fixed large N, then

$$\begin{aligned}(T_N f_N)(\phi) &\geqslant A_N(\phi) - B_N(\phi)\\ &\geqslant k(\sin\phi)^{-1} u(\phi)^{-1/2} \int_0^{c\theta} \left[1 - \frac{K \sin\psi}{k \sin\phi}\right] |f(\psi)|\, u(\psi)^{1/2}\, d\psi\\ &\geqslant k/2(\sin\phi)^{-1} u(\phi)^{-1/2} \int_0^{c\theta} |f(\psi)|\, u(\psi)^{1/2}\, d\psi,\end{aligned}$$

where c is chosen so that $1 - (K\sin\psi)/(k\sin\phi) \geqslant 1 - (K\sin\ \theta)/(k\sin\theta) \geqslant 1/2$.

Now, there exists $C_1 > 0$ such that, on $0 < \phi < \pi/2$, the decreasing function $h(\phi) = (\sin(\phi/2))^{-(\alpha+3/2)}$ satisfies

$$\frac{1}{C_1} h(\phi) \leqslant (\sin\phi)^{-1} u(\phi)^{-1/2} \leqslant C_1 h(\phi).$$

This means

$$\begin{aligned}&\int_J [(\sin\phi)^{-1} u(\phi)^{-1/2} w(\phi)]^p u(\phi)\, d\phi\\ &\leqslant C_1^p \int_J [h(\phi)\, w(\phi)]^p u(\phi)\, d\phi\\ &\leqslant [C_1 h(\theta)]^p \int_J w(\phi)^p u(\phi)\, d\phi\\ &\leqslant [2^{\alpha+3/2} C_1 \inf_{\phi\in J} h(\phi)]^p \int_J w(\phi)^p u(\phi)\, d\phi\\ &\leqslant [2^{\alpha+3/2} C_1]^p \|h\cdot\chi_J\|^p_{L^{p\infty}(w^p u)}\\ &\leqslant [2^{\alpha+3/2} C_1^2]^p \|(\sin(\cdot))^{-1} u^{-1/2} \chi_J\|_{L^{p\infty}(w^p u)}.\end{aligned}$$

Thus,

$$
\begin{aligned}
C \int_0^{c\theta} & |f(\psi)\, w(\psi)|^p\, u(\psi)\, d\psi \\
& \geqslant \| T_N f_N \|^p_{L^{p\infty}(w^p u)} \\
& \geqslant (k/2)^p\, \| (\sin(\cdot))^{-1}\, u^{-1/2} \chi_J \|^p_{L^{p\infty}(w^p u)} \left[\int_0^{c\theta} |f(\psi)|\, u(\psi)^{1/2}\, d\psi \right]^p \\
& \geqslant (k/2)^p\, [2^{\alpha+3/2} C_1^2]^{-p} \int_J [(\sin\phi)^{-1}\, u(\phi)^{-1/2}\, w(\phi)]^p\, u(\phi)\, d\phi \\
& \quad \times \left[\int_0^{c\theta} |f(\psi)|\, u(\psi)^{1/2}\, d\psi \right]^p \\
& \geqslant D^{-1} (k/2)^p\, [2^{\alpha+3/2} C_1^2]^{-p} \int_\theta^{2\theta} w(\phi)^p\, u(\phi)^{1-p/2}\, (\sin\phi)^{-p}\, d\phi \\
& \quad \times \left[\int_0^{c\theta} |f(\psi)|\, u(\psi)^{1/2}\, d\psi \right]^p.
\end{aligned}
$$

Finally, taking the supremum over f such that $\int_0^{c\theta} |f(\psi)\, w(\psi)|^p\, u(\psi)\, d\psi = 1$, we get (36).

IV. PROOF OF THEOREM 3

Sufficiency. Using the estimate (31) for $S_N f$ we need only show

$$
\| J_k(f, \cdot, N) \|_{L^{p\infty}(w^p u)} \leqslant C\, \| f \|_{L^{p1}(w^p u)}, \qquad k = 1, 2, 3, \tag{41}
$$

where $C > 0$ is independent of f and N. The argument in the proof of Theorem 2, though with Hölder's inequality for Lorentz spaces, gives

$$
\begin{aligned}
J_1(f, \theta, N) & \leqslant C[1 + u(\theta)^{-1/2}] \int_0^\pi [1 + u(\phi)^{-1/2}]\, |f(\phi)|\, u(\phi)\, d\phi \\
& \leqslant C[1 + u(\theta)^{-1/2}]\, \| (1 + u^{-1/2})\, w^{-p} \|_{L^{p'\infty}(w^p u)}\, \| f \|_{L^{p1}(w^p u)}.
\end{aligned}
$$

In view of Lemma 7, $P_1 + P_2$ and $Q_1 + Q_2$ are bounded from $L^{p1}(w^p u)$ to $L^{p\infty}(w^p u)$. The former means $\int_0^\pi |f(\phi)|\, u(\phi)^{1/2}\, d\phi < \infty$ for all $f \in L^{p1}(w^p u)$ or, equivalently,

$$
\| u^{-1/2} w^{-p} \|_{L^{p'\infty}(w^p u)} < \infty; \tag{42}
$$

the latter means

$$\|u^{-1/2}\|_{L^{p\infty}(w^p u)} < \infty. \tag{43}$$

Thus,

$$\begin{aligned}\|J_1(f; N)\|_{L^{p\infty}(w^p u)} &\leqslant C\,\|1+u^{-1/2}\|_{L^{p\infty}(w^p u)}\,\|(1+u^{-1/2})\,w^{-p}\|_{L^{p'\infty}(w^p u)}\,\|f\|_{L^{p1}(w^p u)}\\ &\leqslant C\,\|f\|_{L^{p1}(w^p u)},\end{aligned}$$

by (11), (12), (42), and (43).

We next show that, with

$$g(\phi) = b_{N+1}(\phi)\sin\phi f(\phi)\,s_N(\phi)^{-(\alpha+1/2)}\,c_N(\phi)^{-(\beta+1/2)}\,u(\phi),$$

$J_2(f, \theta, N)$ is dominated by a constant multiple of

$$\begin{aligned}&s_N(\theta)^{-(\alpha+1/2)}\left|\int_{\theta/2}^{3\theta/2}\frac{g(\phi)}{\sin((\theta-\phi)/2)}\,d\phi\right| + (M_u f)(\theta) + [(P_1+Q_1)(|f|)](\theta)\\ &\quad + (1+u(\theta)^{-1/2})\int_0^\pi |f(\phi)|\,u(\phi)^{1/2}\,d\phi,\end{aligned} \tag{44}$$

when $0<\theta<\pi/2$, and by

$$\begin{aligned}&c_N(\theta)^{-(\beta+1/2)}\left|\int_{(3\theta-\pi)/2}^{(\theta+\pi)/2}\frac{g(\phi)}{\sin((\theta-\phi)/2)}\,d\phi\right| + (M_u f)(\theta)\\ &\quad + [(P_2+Q_2)(|f|)](\theta)\\ &\quad + (1+u(\theta)^{-1/2})\int_0^\pi |f(\phi)|\,u(\phi)^{1/2}\,d\phi,\end{aligned} \tag{45}$$

when $\pi/2<\theta<\pi$.

Suppose $0<\theta<\pi/2$. As in the proof of Theorem 2,

$$\begin{aligned}J_2(f, \theta, N) &\leqslant C s_N(\theta)^{-(\alpha+1/2)}\,[\,|(Hg)(\theta)| + [(S^1+S^2)(|g|)](\theta)]\\ &\leqslant C s_N(\theta)^{-(\alpha+1/2)}\left[\left|\int_{\theta/2}^{3\theta/2}\frac{g(\phi)}{\sin((\theta-\phi)/2)}\,d\phi\right| + [S_1(|g|)](\theta)\right].\end{aligned}$$

If $\alpha \geqslant -1/2$,

$$\begin{aligned} s_N(\theta)^{-(\alpha+1/2)} & [S^1(|g|)](\theta) \\ & \leqslant C u(\theta)^{-1/2} \left[\left(\sin \frac{\theta}{2} \right)^{-1} \int_0^\theta |f(\phi)|\, u(\phi)^{1/2}\, d\phi \right. \\ & \qquad \left. + \int_\theta^{\pi/2} |f(\phi)|\, u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)} + \int_{\pi/2}^\pi |f(\phi)|\, u(\phi)^{1/2}\, d\phi \right] \\ & \leqslant C \left[[(P_1 + Q_1)(|f|)](\theta) + u(\theta)^{-1/2} \int_0^\pi |f(\phi)|\, u(\phi)^{1/2}\, d\phi \right]. \end{aligned}$$

Again, if $-1 < \alpha < -1/2$,

$$\begin{aligned} [S^1(|g|)](\theta) \leqslant C \Bigg[& N^{\alpha+3/2} \int_0^{1/N} \frac{|f(\phi)| \sin^{2\alpha+2}(\phi/2)}{\sin(\theta/2) + \sin(\phi/2)}\, d\phi \\ & + \int_{1/N}^{\pi/2} \frac{|f(\phi)|\, u(\phi)^{1/2}}{\sin(\theta/2) + \sin(\phi/2)}\, d\phi \\ & + \int_0^\pi |f(\phi)|\, u(\phi)^{1/2}\, d\phi \Bigg]. \end{aligned}$$

But,

$$s_N(\theta)^{-(\alpha+1/2)} \int_0^\pi |f(\phi)|\, u(\phi)^{1/2}\, d\phi \leqslant C[1 + u(\theta)^{-1/2}] \int_0^\pi |f(\phi)|\, u(\phi)^{1/2}\, d\phi.$$

Moreover, when $0 < \theta < 1/N$,

$$\begin{aligned} S_N(\theta)^{-(\alpha+1/2)} N^{\alpha+3/2} & \int_0^{1/N} \frac{|f(\phi)| \sin^{2\alpha+2}(\phi/2)}{\sin(\theta/2) + \sin(\phi/2)}\, d\phi \\ & \leqslant C N^{2\alpha+2} \int_0^{1/N} |f(\phi)|\, u(\phi)\, d\phi \\ & \leqslant C(M_u f)(\theta) \end{aligned}$$

and

$$\begin{aligned} S_N(\theta)^{-(\alpha+1/2)} & \int_{1/N}^{\pi/2} \frac{|f(\phi)|\, u(\phi)^{1/2}}{\sin(\theta/2) + \sin(\phi/2)}\, d\phi \\ & \leqslant C N^{\alpha+1/2} \int_{1/N}^{\pi/2} |f(\phi)|\, u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)} \end{aligned}$$

$$\leqslant CN^{\alpha+1/2} \sum_{k=1}^{\log_2(N\pi/2)} \int_{2^{k-1}/N}^{2^k/N} |f(\phi)|\, u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)}$$

$$\leqslant C \sum_{k=1}^{\log_2(N\pi/2)} 2^{(\alpha+1/2)k} \left(\frac{N}{2^k}\right)^{2\alpha+2} \int_\theta^{2k/N} |f(\phi)|\, u(\phi)\, d\phi$$

$$\leqslant C(M_u f)(\theta);$$

when $1/N < \theta < \pi/2$,

$$S_N(\theta)^{-(\alpha+1/2)} N^{\alpha+3/2} \int_0^{1/N} \frac{|f(\phi)| \sin^{2\alpha+2}(\phi/2)}{\sin(\theta/2)+\sin(\phi/2)}\, d\phi$$

$$\leqslant C\left(\sin\frac{\theta}{2}\right)^{-1} u(\theta)^{-1/2} \int_0^\theta |f(\phi)|\, (N\phi)^{\alpha+3/2}\, u(\phi)^{1/2}\, d\phi$$

$$\leqslant C[P_1(|f|)](\theta),$$

and

$$s_N(\theta)^{-(\alpha+1/2)} \int_{1/N}^{\pi/2} \frac{|f(\phi)|\, u(\phi)^{1/2}}{\sin(\theta/2)+\sin(\phi/2)}\, d\phi$$

$$\leqslant C\left[\left(\sin\frac{\theta}{2}\right)^{-1} u(\theta)^{-1/2} \int_{1/N}^\theta |f(\phi)|\, u(\phi)^{1/2}\, d\phi \right.$$

$$\left. + u(\theta)^{-1/2} \int_0^{\pi/2} |f(\phi)|\, u(\phi)^{1/2} \frac{d\phi}{\sin(\phi/2)}\right]$$

$$\leqslant C[(P_1+Q_1)(|f|)](\theta).$$

The asserted upper bound for $J_2(f, \theta, N)$, $\pi/2 < \theta < \pi$, is obtained in the same way.

We now prove (41) for $k=2$ using (44) and (45); the proof for $k=3$ is similar. We have

$$\|M_u f + (P_1+Q_1+P_2+Q_2)\, f\|_{L^{p\infty}(w^p u)} \leqslant C\, \|f\|_{L^{p1}(w^p u)},$$

by (15)–(19), while

$$\left\|(1+u^{-1/2}) \int_0^\pi |f(\phi)|\, u(\phi)^{1/2}\, d\phi\right\|_{L^{p\infty}(w^p u)} \leqslant c\, \|f\|_{L^{p1}(w^p u)}$$

has been shown in the course of proving (40) for $k=1$.

To obtain

$$\left\| s_N(\theta)^{-(\alpha+1/2)} \left| \int_{\theta/2}^{3\theta/2} \frac{g(\phi)}{\sin((\theta-\phi)/2)}\, d\phi \right| \chi_{(0,\,\pi/2)}(\phi) \right\|_{L^{p\infty}(w_p u)} \leqslant C\, \|f\|_{L^{p1}(w^p u)},$$

we again take advantage of the fact that it suffices to consider $f=\chi_E$, $E\subset(0,\pi/2)$. We write $g_k=g\cdot\chi_{I_k}$, $I_k=(\pi/2^{k+2}, 3\pi/2^{k+1})$, $k=1, 2, ...,$ whence

$$\int_{\theta/2}^{3\theta/2} \frac{g(\phi)}{\sin((\theta-\phi)/2)}\, d\phi\cdot\chi_{(0,\,\pi/2)}(\theta)=\sum_{k=1}^{\infty} (Hg_k)(\theta)\cdot\chi_{J_k}(\theta),$$

$J_k=(\pi/2^{k+1}, \pi/2^k)$. Therefore,

$$\lambda^p \int_{H_\lambda} w(\theta)^p\, u(\theta)\, d\theta=\sum_{k=1}^{\infty} \lambda^p \int_{H_\lambda^k} w(\theta)^p\, u(\theta)\, d\theta,$$

where

$$H_\lambda=\left\{\theta\in(0,\pi/2)\colon s_N(\theta)^{-(\alpha+1/2)}\left|\int_{\theta/2}^{3\theta/2} \frac{g(\phi)}{\sin((\theta-\phi)/2)}\, d\phi\right|>\lambda\right\}$$

and

$$H_\lambda^k=\{\theta\in J_k\colon s_N(\theta)^{-(\alpha+1/2)}\, |(Hg_k)(\theta)|>\lambda\}.$$

At this point we observe that $w^p u$ satisfies the A_∞ condition

$$\frac{|E|}{|I|}\leqslant C\left[\frac{\int_E w(\theta)^p\, u(\theta)\, d\theta}{\int_I w(\theta)^p\, u(\theta)\, d\theta}\right]^r,$$

with $C>0$ independent of the interval $I\subset(0,\pi)$ and the measurable set $E\subset I$, in view of (15) and the fact that u satisfies A_∞. The arguments of [21, Chapter XIII] then yield $C>0$ for which

$$\begin{aligned} \|Hg_k\|_{L^{p\infty}(w^p u)} &\leqslant C\, \|Mg_k\|_{L^{p\infty}(w^p u)} \\ &\leqslant C\, \|M_u g_k\|_{L^{p\infty}(w^p u)}, \end{aligned} \tag{46}$$

$k=1, 2, ...$.

Letting

$$G_\lambda^k=\{\theta\in(0,\pi/2)\colon (Hg_k)(\theta)>\lambda_k\},$$

$\lambda_k = \min[s_N(\pi/2^k), s_N(\pi/2^{k+1})]^{\alpha+1/2} \lambda = c_k \lambda$, we have

$$\begin{aligned}\lambda^p \int_{H_\lambda^k} w(\theta)^p\, u(\theta)\, d\theta &\leqslant \lambda^p \int_{G_\lambda^k} w(\theta)^p\, u(\theta)\, d\theta \\ &\leqslant c_k^{-p}\, \|Hg_k\|^p_{L^{p\infty}(w^p u)} \\ &\leqslant Cc_k^{-p}\, \|M_u g_k\|^p_{L^{p\infty}(w^p u)} \qquad \text{by (46)} \\ &\leqslant Cc_k^{-p}\, \|g_k\|^p_{L^{p1}(w^p u)} \qquad \text{by (15)} \\ &\leqslant C \int_{E\cap I_k} w(\theta)^p\, u(\theta)\, d\theta, \qquad k = 1, 2, \ldots .\end{aligned}$$

Thus,

$$\left\| s_N(\theta)^{-(\alpha+1/2)} \left| \int_{\theta/2}^{3\theta/2} \frac{g(\phi)}{\sin((\theta-\phi)/2)}\, d\phi \right| \lambda_{(0,\, \pi/2)}(\theta) \right\|^p_{L^{p\infty}(w^p u)}$$

$$\leqslant C \sum_{k=1}^{\infty} \int_{E\cap I_k} w(\theta)^p\, u(\theta)\, d\theta = C \int_E w(\theta)^p\, u(\theta)\, d\theta = C\, \|\chi_E\|^p_{L^{p1}(w^p u)}$$

and we are done on taking pth roots.

Necessity. First we observe (15) follows from the consequence

$$\|f(r, \cdot)\|_{L^{p\infty}(w^p u)} \leqslant C\, \|f\|_{L^{p1}(w^p u)}$$

of (8), together with (34), on using the argument of Proposition 1 in [8].

Next, we show (8) implies (17) to illustrate the method of proving the necessity of the remaining conditions. To this end, fix $\theta \in (0, \pi/2B)$ and $E \subset (\theta, \pi/2)$, E a finite disjoint union of intervals, with $B > 3/2$ to be specified below. Consider functions $0 \leqslant f = f \cdot \chi_E$ and define T_N and f_N as in (39) and (40), respectively. By (8), we have

$$\|T_N g\|_{L^{p\infty}(w^p u)} \leqslant C\, \|g\|_{L^{p1}(w^p u)}, \qquad g \in L^{p1}(w^p u);$$

further

$$\begin{aligned}|(T_N f_N)(\phi)| &\geqslant |p_{N+1}(\cos\phi)| \int_E \frac{|q_N(\cos\psi)|\, f(\psi)\, u(\psi)}{\sin((\psi+\phi)/2)\sin((\psi-\phi)/2)}\, d\psi \\ &\quad - |q_N(\cos\theta)| \int_E \frac{|p_{N+1}(\cos\psi)|\, f(\psi)\, u(\psi)}{\sin((\psi+\phi)/2)\sin((\psi-\phi)/2)}\, d\psi \\ &\geqslant C_N(\phi) - D_N(\phi) \qquad \phi \in (0, \theta).\end{aligned}$$

Arguing as in the proof of Theorem 2, we obtain, for a sufficiently large N (in particular, $N > 2/B\theta$) two sets of pairwise disjoint intervals $I_\ell = I_\ell(N) \subset (1/BN, \theta)$, $\ell = 1, 2, ..., \ell_0 = \ell_0(N)$ and $K_m = K_m(N) \subset E$, $m = 1, 2, ..., m_0 = m_0(N)$, such that $\bigcup_{\ell=1}^{\ell_0} 3I_\ell \supset (1/BN, \theta)$, $\bigcup_{m=1}^{m_0} 3K_m \supset E$, $3I_\ell$, $3K_m$ intervals of type (ii)[1] ; moreover, when $\phi \in F = \bigcup_{\ell=1}^{\ell_0} I_\ell$, $\psi \in G = \bigcup_{m=1}^{m_0} K_m$,

$$|p_{N+1}(\cos\phi)| \geqslant \frac{1}{3}\sqrt{\frac{\pi}{2}}\, 2^{\alpha+\beta} \sin^{-(\alpha+1/2)}\frac{\phi}{2} \cos^{-(\beta+1/2)}\frac{\phi}{2}$$

and

$$|p_{N+1}^{(\alpha+1,\beta+1)}(\cos\psi)| \geqslant \frac{1}{3}\sqrt{\frac{\pi}{2}}\, 2^{\alpha+\beta+2} \sin^{-(\alpha+3/2)}\frac{\psi}{2} \cos^{-(\beta+3/2)}\frac{\psi}{2}.$$

Restricting attention to $f = f \cdot \chi_G$, the latter yield

$$\begin{aligned} c_N(\phi) &\geqslant k \sin^{-(\alpha+1/2)}\frac{\phi}{2} \int_G \sin^2\psi\, |p_N^{(\alpha+1,\beta+1)}(\cos\psi)|\, f(\psi)(\sin\psi)^{-2} \\ &\quad \times \sin^{2(\alpha+1)+1}\frac{\psi}{2} \cos^{2(\beta+1)}\frac{\psi}{2}\, d\psi \\ &\geqslant k \sin^{-(\alpha+1/2)}\frac{\phi}{2} \int_G f(\psi) \sin^{\alpha-1/2}\frac{\psi}{2}\, d\psi, \qquad \phi \in F, \end{aligned}$$

which, together with

$$D_N(\phi) \leqslant K \sin^{-(\alpha+1/2)}\frac{\phi}{2} \int_G f(\psi) \sin^{\alpha-1/2}\frac{\psi}{2}\, d\psi,$$

gives

$$\begin{aligned} (T_N f_N)(\theta) &\geqslant k \sin^{-(\alpha+1/2)}\frac{\phi}{2} \int_G f(\psi) \sin^{\alpha-1/2}\frac{\phi}{2}\left[1 - \frac{K}{k}\frac{\phi}{\psi}\right] d\psi \\ &\geqslant k/2 \sin^{-(\alpha+1/2)}\frac{\phi}{2} \int_G f(\psi) \sin^{\alpha-1/2}\frac{\psi}{2}\, d\psi \qquad \psi \in F, \end{aligned}$$

when $(K/k) \cdot (1/B) \leqslant 2$ or $B \geqslant 2K/k$. Thus, choosing $B = 2K/k + 3/2$,

$$\begin{aligned} C\, \|f \cdot \chi_G\|_{L^{p_1}(w^p u)} &\geqslant \|T_N f_N \cdot \chi_F\|_{L^{p_\infty}(w^p u)} \\ &\geqslant \frac{k}{2} \left\| \sin^{-(\alpha+1/2)}\frac{\phi}{2} \cdot \chi_F(\phi) \right\|_{L^{p_\infty}(w^p u)} \int_G f(\psi) \sin^{\alpha-1/2}\frac{\psi}{2}\, d\psi. \end{aligned}$$

[1] See page 19.

Taking the supremum over f with $\|f\cdot\chi_G\|_{L^{p1}(w^p u)}=1$, we obtain

$$\|u(\phi)^{-1/2}\chi_F(\phi)\|_{L^{p\infty}(w^p u)}\left\|\frac{u(\phi)^{-1/2}}{\sin(\phi/2)}w(\phi)^{-p}\chi_G(\phi)\right\|_{L^{p'\infty}(w^p u)}\leqslant C,$$

which means

$$\left[\int_H w(\phi)^p u(\phi)\,d\phi\right]^{1/p-1}\int_H u(\phi)^{1/2} w(\phi)^p\,d\phi\left[\int_G w(\phi)^p u(\phi)\,d\phi\right]^{-1}$$
$$\times\int_G u(\phi)^{1/2}\frac{d\phi}{\sin(\phi/2)}\leqslant C \tag{47}$$

for all $H\subset F$. In particular, set $H=(\theta/2,\theta)\cap F$ and observe that

$$\int_{\theta/2}^{\theta} w(\phi)^p u(\phi)\,d\phi\leqslant C\int_H w(\phi)^p u(\phi)\,d\phi,$$

by (15), whence (47) implies

$$u(\theta)^{-1/2}\int_G u(\phi)^{1/2}\frac{d\phi}{\sin(\phi/2)}\leqslant C\left[\frac{\int_G w(\phi)^p u(\phi)\,d\phi}{\int_{\theta/2}^{\theta} w(\phi)^p u(\phi)\,d\phi}\right]^{1/p}$$

and so (17), since (15) ensures $w^p u$ is doubling, as is $u(\phi)^{1/2}/\sin(\phi/2)$.

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